On the propagation of weak shock waves

By G. B. WHITHAM
*Department of Mathematics, University of Manchester**

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SUMMARY

A method is presented for treating problems of the propagation and ultimate decay of the shocks produced by explosions and by bodies in supersonic flight. The theory is restricted to weak shocks, but is of quite general application within that limitation. In the author's earlier work on this subject (Whitham 1952), only problems having directional symmetry were considered; thus, steady supersonic flow past an axisymmetrical body was a typical example. The present paper extends the method to problems lacking such symmetry. The main step required in the extension is described in the introduction and the general theory is completed in § 2; the remainder of the paper is devoted to applications of the theory in specific cases.

First, in § 3, the problem of the outward propagation of spherical shocks is reconsidered since it provides the simplest illustration of the ideas developed in § 2. Then, in § 4, the theory is applied to a model of an unsymmetrical explosion. In § 5, a brief outline is given of the theory developed by Rao (1956) for the application to a supersonic projectile moving with varying speed and direction. Examples of steady supersonic flow past unsymmetrical bodies are discussed in §§ 6 and 7. The first is the flow past a flat plate delta wing at small incidence to the stream, with leading edges swept inside the Mach cone ; the results agree with those previously found by Lighthill (1949) in his work on shocks in cone field problems, and this provides a valuable check on the theory. The second application in steady supersonic flow is to the problem of a thin wing having a finite curved leading edge. It is found that in any given direction the shock from the leading edge ultimately decays exactly as for the bow shock on a body of revolution; the equivalent body of revolution for any direction is determined in terms of the thickness distribution of the wing and varies with the direction chosen. Finally in § 8, the wave drag on the wing is calculated from the rate of dissipation of energy by the shocks. The drag is found to be the mean of the drags on the equivalent bodies of revolution for the different directions.

* Now at Institute of Mathematical Sciences, New York University.

1. Introduction

In a previous paper (Whitham 1952), a method was developed for determining the symmetrical shocks which are produced, for example, by an axisymmetric body in steady supersonic flow or by a spherically symmetric explosion. Due to the directional symmetry in these problems, the flow quantities depend only on two independent variables. In this paper, the method is extended to deal with problems which involve more than two independent variables. For shocks produced by bodies in supersonic flight, such problems arise when the body is unsymmetrical or when the velocity is not uniform; they also arise in explosion problems when the initial shape of charge is not spherical.

The theory is restricted to weak shocks and for that reason the applications to explosion problems will be of practical value only at distances from the explosion which are sufficiently large for the shocks to be weak. However, in formulating the basic ideas of the theory, it is convenient to start with the problem of a weak explosion and, in fact, to consider the following simplified model. A region of arbitrary shape, bounded by a surface S, contains gas at a pressure higher than that of its surroundings; initially the gas is at rest with uniform pressure but at time $t = 0$ it is suddenly released. According to the theory of sound, the wavefront carrying the first disturbance outwards from the explosion moves along the normals to the surface S with velocity a_0, where a_0 is the constant sound speed in the undisturbed gas surrounding the explosion. These normals are the orthogonal trajectories of the successive positions of the wavefront and are known as 'rays'; in a sense, the rays are the carriers of the disturbance. Moreover, the appropriate solution to this problem in the theory of sound predicts the magnitude of the disturbance, and in particular the variation in the magnitude of the pressure jump at the wavefront as it moves out along a ray. However, near the head of the wave, the law of variation of the amplitude of the disturbance takes a simple form which can be deduced quite generally from the approximation of 'geometrical acoustics', without appeal to the detailed solution in the full theory. For, in certain circumstances, the energy propagated down a narrow ray tube formed by a bundle of neighbouring rays is conserved; that is, reflection and diffraction of energy may be neglected. Hence, since the flux of energy across any section of the ray tube is proportional to the square of the amplitude multiplied by the cross-sectional area of the tube, the amplitude varies with distance s along the tube like $A^{-1/2}(s)$, where $A(s)$ is proportional to the cross-sectional area at the point s.

But, even when there is no reflection or diffraction of energy, the dissipation of energy by the shock wave (which in reality replaces the wavefront) and the related distortion of the wave profile behind the shock due to non-linear effects cannot be ignored. Thus, even for weak shocks, the result that the shock strength varies with s like $A^{-1/2}(s)$ requires modification, just as in the symmetric problems of the original theory, It should be stressed that this inaccuracy is a failure of the linear theory of sound and is not introduced by the approximations of geometrical acoustics.

Now, although its prediction of the shock strength is incorrect, geometrical acoustics provides the key to the solution of these more general shock problems: *we assume for them also that the propagation of the disturbance down each ray tube may be treated separately*. This gives a two variable problem depending on time t and distance s, and it can be solved by precisely those methods which were developed in the original theory. The other variables in the problem appear only as parameters in the function $A(s)$ and in the function which specifies the initial wave profile for each tube.

In the improved theory, any point of a shock moves with the speed appropriate to the strength of the shock at that point. Thus, even for propagation into a uniform medium, if the strength varies along the shock there will be a tendency for the shock to be refracted away from the wavefront positions given by the linear theory (which assumes the uniform speed of propagation a_0). As a consequence, the true orthogonal trajectories of the shock positions will curve away from the straight rays of the linear theory. In principle, therefore, the ray tubes need modification at the same time as the modification to the law of propagation in each tube. However, unless the strength varies very rapidly along the shock, the effect of the curvature of the rays is relatively small. The displacement of the ray from its linear position may become large as $s \to \infty$, but the displacement remains small compared to s, and the total angle turned by the ray is small. Since we are most interested in the directional distribution of shock strength for given s, this error may usually be neglected.

Nevertheless, situations do arise in which the shock strength varies rapidly along the shock. An example of this occurs in the explosion problem when the surface S is concave outwards in some region. For, then, the rays intersect, and since $A \to 0$ at points of intersection, the linear theory predicts that the strength will become infinite. Of course, the linear theory breaks down completely and the consequences are shown in figure 1; AB is the initial surface, CD, EFG, $HIJK$ are successive positions of the wavefront and the dotted line is the 'caustic', i.e. the envelope of the rays. Now, when the properties of a real shock are taken into account, this singular behaviour is avoided. In the concave part, due to the convergence of the flow, the shock increases in strength and therefore moves faster than the neighbouring parts. This effect smooths out the concavity and the rays curve away from each other, avoiding intersections in the neighbourhood of F. The true type of behaviour is sketched in figure 2. In such a case the distortion of the rays is crucial. By making simplifying assumptions, this distortion can be studied to some extent mathematically and a theory can be given which covers the main features of figure 2. However, the theory forms part of a separate investigation which has more general applications; accordingly, it will be left to a later paper. In most of the problems treated here, this question does not arise, and it will be sufficient, for the overall picture, to treat only the propagation outside such singular regions, bearing in mind the qualitative description of the behaviour as represented in figure 2. Thus, the theory will be presented neglecting the deviation of the rays from their linear positions.

So far the basic notions have been described for the explosion problem, but the account applies in general to all the problems to be considered, provided that in supersonic flow problems the system of reference is so chosen that the air is at rest at infinity. In each case the linear theory (based on the assumption of small disturbances) represents the shock as

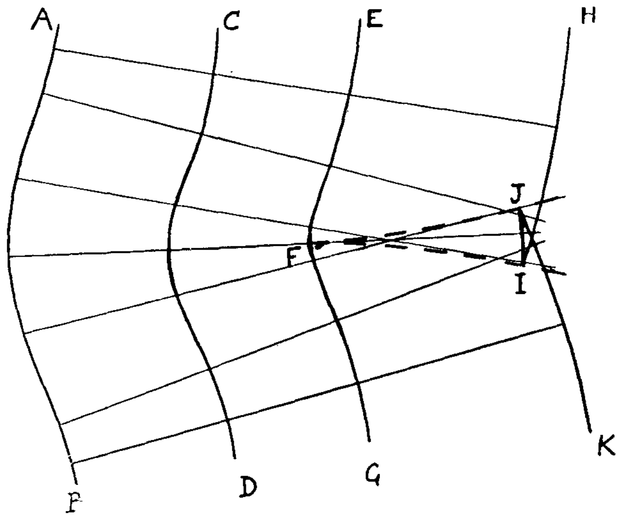

Figure 1.

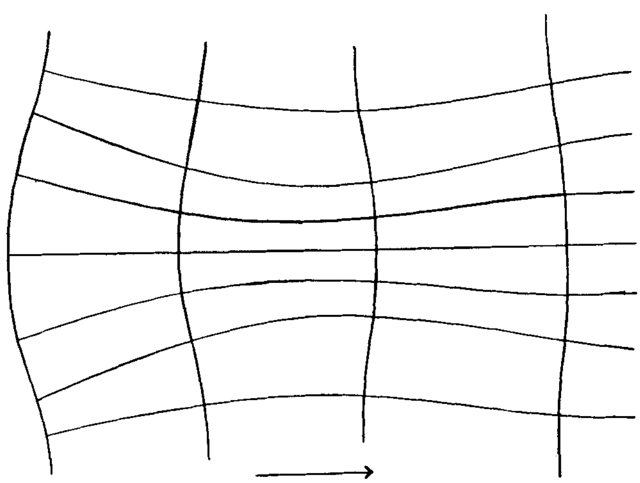

Figure 2.

a wavefront spreading out from the body or explosion with its amplitude varying as $A^{-1/2}(s)$, where $A(s)$ is proportional to the area of the appropriate ray tube, and in deducing the correct results, the flow in each ray tube may be considered separately. Thus, when the general results for propagation in a non-uniform tube have been established, the application to specific problems reduces to the determination of the area function $A(s)$ and the initial wave profile for each ray tube. In those problems of supersonic flow which reduce to steady flow problems when axes fixed with respect to the body are chosen, it is more convenient to treat the steady flow problem directly rather than to introduce the time as an additional variable. The ideas developed above have similar interpretations in steady flow; these will be given and used in §§ 6 and 7 for examples from that field.

Only propagation through a uniform medium will be considered, but it may be noted that this is not an essential restriction. The theory of geometrical acoustics will always give the basic geometrical picture of wavefronts and rays, but when the sound speed is not constant the rays curve round, as the wavefront is refracted. The prediction given by the theory of sound for the amplitude of the distrubance may again be deduced from energy conservation along a narrow ray tube, but factors involving the density ρ_0 and sound speed a_0 in the undisturbed fluid are no longer constant and must be retained in the expression for the energy flux. For example, the amplitude of the pressure variation is proportional to $(\rho_0 a_0)^{1/2}A^{-1/2}$. In principle, the corrected results for the propagation of the shocks can be deduced by the methods described below.

2. General theory

In this section, a general account is given of the method for improving the linear theory of the propagation in individual ray tubes. The method is developed from physical arguments; but, as a check, the results for the ultimate decay of the disturbance at large distances along the ray tube are also deduced mathematically. The required non-linear features may be introduced by taking account of the progressive distortion of the wave profile due to the small variations in the values of the propagation speeds of the individual wavelets in the wave; each wavelet travels with the local speed of sound a relative to the fluid, and this is only approximately equal to a_0. The dissipation of energy at the shock is then incorporated automatically at a later stage, simply by applying the Rankine–Hugoniot shock conditions.

The first step is to examine in more detail the results given by the theory of sound. For the most part, problems of shocks moving into undisturbed fluid are considered, and therefore it is appropriate to find the expressions for the flow quantities near the head of the wave. Now when the various problems are considered, it is found in general that in each ray tube the pressure increment $p-p_0$ and particle velocity u are proportional to

$$\frac{F(t-s/a_0)}{\sqrt{A(s)}}, \tag{1}$$

where $A(s)$ is the ray tube area and the function F, which determines the detailed wave profile, depends on the initial conditions in the particular problem considered. Thus, near the head of the wave, the amplitude is correctly predicted by geometrical acoustics; however, the full solution has to be used in order to determine the function F.

To include the non-linear distortion of the wave profile, (1) is modified to

$$\frac{F(\tau)}{\sqrt{A}}, \tag{2}$$

where $\tau(t, s)$ is to be determined so that each wavelet specified by τ = constant travels with the accurate speed $a+u$ in place of a_0. Hence, τ is to be determined from

$$\left(\frac{ds}{dt}\right)_{\tau = \text{constant}} = a+u. \tag{3}$$

Since the disturbance is assumed to be small, $a-a_0$ is proportional to $p-p_0$; therefore, $a+u$ differs from a_0 by a multiple of $F(\tau)/A$. Then, to the same order of approximation, (3) may be written

$$\left(\frac{dt}{ds}\right)_{\tau = \text{constant}} = \frac{1}{a+u} \doteqdot \frac{1}{a_0} - \frac{kF(\tau)}{A}, \tag{4}$$

where k is a constant. The arbitrary function of τ which arises in the integration is fixed so that τ takes the linear value $t-s/a_0$ near $s = 0$, in order that the wave profile agrees with (1) initially. Then, (4) gives

$$t = \frac{s}{a_0} - kF(\tau)\int_0^s \frac{ds}{\sqrt{A}} + \tau, \tag{5}$$

and the increasing divergence between the values of τ and $t-s/a_0$ as s increases gives the progressive distortion of the wave profile.

In all the cases considered, it is found that $u \doteqdot (p-p_0)/\rho_0 a_0$, where ρ_0 is the density of the undisturbed gas (the necessity for this result is seen below in equation (10)), and for a polytropic equation of state ($p \propto \rho^\gamma$)

$$\frac{a-a_0}{a_0} = \frac{\gamma-1}{2\gamma}\frac{p-p_0}{p_0}. \tag{6}$$

Hence, if $F(\tau)$ is chosen so that

$$\frac{p-p_0}{p_0} = \frac{F(\tau)}{\sqrt{A}}, \tag{7}$$

then

$$\frac{u}{a_0} = \frac{1}{\gamma}\frac{F(\tau)}{\sqrt{A}}, \quad \frac{a-a_0}{a_0} = \frac{\gamma-1}{2\gamma}\frac{F(\tau)}{\sqrt{A}}, \tag{8}$$

and, in (4),

$$k = \frac{\gamma+1}{2\gamma a_0}. \tag{9}$$

The improved solution to the problem is now given by (7) and (8), with τ given implicitly as a function of t and s by (5). In the problems considered here, the wave is headed by a shock; therefore, to complete the solution, the position and strength of this shock must be determined in accordance with the Rankine–Hugoniot conditions. For a shock moving into undisturbed gas, it may be shown that, to a first order of approximation,

$$\frac{u_1}{a_0} = \frac{1}{\gamma}\frac{p_1-p_0}{p_0}, \quad \frac{a_1-a_0}{a_0} = \frac{\gamma-1}{2\gamma}\frac{p_1-p_0}{p_0}, \quad \frac{U}{a_0} = 1+\frac{\gamma+1}{4\gamma}\frac{p_1-p_0}{p_0}, \tag{10}$$

where U is the shock velocity and the subscript 1 denotes values behind the shock. The first two of these equations are satisfied already in the solution (7) and (8) (even in the linear theory these conditions are satisfied by discontinuities in the flow quantities), and the third condition determines the shock. It may also be noted that the shock conditions show that the entropy jump at the shock is third order in the shock strength, which explains why the entropy changes can be neglected in the flow behind the shock; this point is discussed in detail in Whitham (1952).

Now, the shock is intimately connected with the distortion of the wave profile. Wavelets are continually fed into the shock, where their energy is dissipated. Without the shock, the wave profile would 'break', exactly as in the well-known theory of plane waves of finite amplitude, due to the higher propagation speed of the wavelets in the regions of higher pressure. Thus, the compression regions of the initial wave profile shown in figure 3(i) would eventually break, as shown by the full line curve in figure 3(ii). This

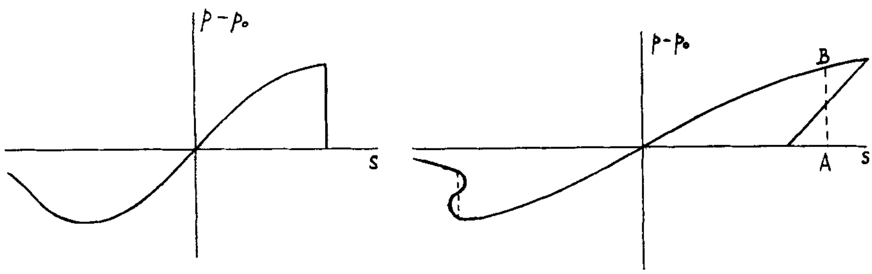

Figure 3.

leads to a solution which is physically unreal since it predicts more than one value of the pressure at certain points. However, shocks cut out the overlapping wavelets as shown by the dotted lines in figure 3(ii); for example, at the head of the wave the pressure jumps almost discontinuously from A to B.

It is clear that the position and strength of the shock are determined if the value of τ corresponding to the flow just behind the shock is known; that is, if the value of τ at the point B on the wave profile is known. For, denoting this value of τ by $T(s)$, the time t at which the shock reaches the position s is found by substituting $\tau = T(s)$ in (5), and the pressure, particle velocity etc., are found by making this substitution in (7) and (8). However, these results must conform with the shock conditions (10), and this determines the function $T(s)$. It follows from (5) that

$$\frac{1}{U} = \frac{dt}{ds} = \frac{1}{a_0} - \frac{kF(T)}{\sqrt{A}} + \left(1 - kF'(T)\int_0^s \frac{ds}{\sqrt{A}}\right)\frac{dT}{ds},$$

and, according to (10), (again retaining only the first order term), this must be equal to

$$\frac{1}{a_0}\left(1 - \frac{\gamma+1}{4\gamma}\frac{p-p_0}{p_0}\right) = \frac{1}{a_0} - \frac{kF(T)}{2\sqrt{A}}.$$

Hence,

$$kF'(T)\left(\int_0^s \frac{ds}{\sqrt{A}}\right)\frac{dT}{ds} + \frac{1}{2}\,\frac{kF(T)}{\sqrt{A}} = \frac{dT}{ds}.$$

After multiplication by 2 $F(T)$, this equation integrates to

$$\int_0^s \frac{ds}{\sqrt{A}} = \frac{2\int_0^T F(T')\,dT'}{kF^2(T)}, \tag{11}$$

assuming for definiteness that $T = 0$ at $s = 0$.

It is found in general that the positive phase of the wave $(p-p_0 > 0)$ in the region immediately behind the shock, is followed by a negative phase $(p-p_0 < 0)$. When this is the case, $F(\tau)$ has a zero for some finite value T_0 of τ and the results for the ultimate decay of the shock take a simple form. For, assuming that $\int_0^s A^{-1/2}\,ds \to \infty$ as $s \to \infty$, (11) shows that $T(s) \to T_0$ as $s \to \infty$; and, in fact, (11) may be approximated for large s as

$$F(T) \sim \left\{\frac{2}{k}\int_0^{T_0} F(T')\,dT'\right\}^{1/2}\left\{\int_0^s \frac{ds}{\sqrt{A}}\right\}^{-1/2} \tag{12}$$

Hence, from (7) and (9), the value of the pressure behind the shock is given by

$$\frac{p_1-p_0}{p_0} \sim \left\{\frac{4\gamma a_0}{\gamma+1}\int_0^{T_0} F(T')\,dT'\right\}^{1/2}\left\{A\int_0^s \frac{ds}{\sqrt{A}}\right\}^{-1/2}, \tag{13}$$

and u_1 and $(a_1-a_0)/a_0$ are proportional to this. From (5), the position of the shock at time t is given by

$$t = \frac{s}{a_0} - \left\{\frac{\gamma+1}{\gamma a_0}\int_0^{T_0} F(T')\,dT'\right\}^{1/2}\left\{\int_0^s \frac{ds}{\sqrt{A}}\right\}^{1/2} + T_0. \tag{14}$$

Thus, at s the shock is ahead of the wavelet $t = T_0 + s/a_0$, on which the pressure is zero, by an amount

$$l(s) \propto \left\{\int_0^s \frac{ds}{\sqrt{A}}\right\}^{1/2}. \tag{15}$$

The quantity l is a measure of the length of the wave and (15) shows how it ultimately increases with s (in contrast to the constant length in the linear theory). The pressure distribution within the wave also takes a simple form. For, at all points τ is near T_0; therefore, equation (5) which determines $\tau(t, s)$ may be approximated as

$$t = \frac{s}{a_0} - kF(\tau)\int_0^s \frac{ds}{\sqrt{A}} + T_0.$$

Hence, using this value for $F(\tau)$, (7) becomes

$$\frac{p-p_0}{p_0} = -\frac{t-T_0-s/a_0}{k\sqrt{A}}\int_0^s \frac{ds}{\sqrt{A}}. \tag{16}$$

Thus, at any point s, the pressure falls ***linearly*** with time and the rate of fall is

$$\frac{\partial}{\partial t}\left(\frac{p-p_0}{p_0}\right) = -\frac{2\gamma}{\gamma+1}a_0\left\{\sqrt{A}\int_0^s \frac{ds}{\sqrt{A}}\right\}^{-1} \tag{17}$$

It is of interest to note that (17) depends only on the distance s and the properties of the fluid; it is independent of the initial wave form given by $F(\tau)$.

An essential point must be made in connection with these results. The expression (1) was originally introduced as the appropriate approximation to the solution given by the theory of sound for the flow quantities near the head of the wave. In the improved theory, wavelets are continually fed into the shock; hence, values of the flow quantities at points away from the head of the wave in the linear theory must also be considered. However, the wavelets are fed into the shock at a relatively slow rate and by the time this effect has become appreciable the wave has travelled a relatively large distance. It is found that (1) again applies for large s even when $t-s/a_0$ is not small (in fact the precise condition for the approximation (1) is usually that $(a_0t-s)/s$ should be small); hence its use is justified throughout the motion of the shock.

The results given by (13) and (15) for the important cases of plane, cylindrical and spherical waves may be singled out for special note. For plane waves,

$$A = \text{constant}, \qquad (p_1-p_0)/p_0 \propto s^{-1/2}, \qquad l \propto s^{1/2}, \tag{18}$$

for cylindrical waves,

$$A \propto s, \qquad (p_1-p_0)/p_0 \propto s^{-3/4}, \qquad l \propto s^{1/4}, \tag{19}$$

and for spherical waves,

$$A \propto s^2, \qquad (p_1-p_0)/p_0 \propto s^{-1}(\log s)^{-1/2}, \qquad l \propto (\log s)^{1/2}. \tag{20}$$

Using the simplified results (13) and (15), it is possible to see the significance of the existence of a zero of $F(\tau)$. First, consider the outward flux of mass at the point s as the wave passes that point. Clearly, the flux due to the positive phase of the wave, in which $F(\tau)>0$, is proportional to $\rho_0 u_1 A l$; according to (13) and (15), this quantity varies as $\sqrt{A}$. But apart from the case of plane waves for which A is constant, it is generally true that $\sqrt{A} \to \infty$ as $s \to \infty$. In the latter case, therefore, it is impossible for the wave to have only a positive phase, and so it must be followed by a negative phase; hence $F(\tau)$ has a zero. Furthermore, in any realistic case, the fluid returns to a state of rest after the whole wave passes to that there is a recompression region after the negative phase, and a second shock must ultimately be formed (due to the distortion and 'breaking' of the wave profile). Since the positive and negative phases of the wave must eventually produce equal and opposite contributions to the mass flux, the wave profile will tend to a symmetrical N-wave in which the two shocks have equal strengths and the rate of fall of p between the shocks is given by (17). The formation and subsequent motion of the second shock can be described in detail by the present theory, and the tendency to form a symmetrical N-wave is confirmed. A full account of this is given in Whitham (1952) for the special cases of steady supersonic flow past an axisymmetric body and unsteady plane waves; the extensions to the more general case considered here are trivial.

At this stage, having described the main structure of the theory, it is suitable to reconsider the physical assumptions that have been made, and to give mathematical checks where possible. There are two points which require further verification. The first concerns the energy dissipation by the shock; this was not introduced explicitly and the question arises of whether it is correctly accounted for by making use of the shock conditions. Now, it is easily verified from the results in (13) and (15) that the wave loses energy at exactly the same rate as the shocks dissipate energy. The full details will not be given, but we may note that the two quantities have the same dependence on s. The energy carried by the wave is proportional to

$$\rho_0 a_0^2 A l \left(\frac{p-p_0}{p_0}\right)^2;$$

from (13) and (15), this quantity varies with s like

$$\left\{\int_0^s \frac{ds}{\sqrt{A}}\right\}^{-1/2},$$

and it decreases at a rate proportional to

$$\frac{1}{\sqrt{A}}\left\{\int_0^s \frac{ds}{\sqrt{A}}\right\}^{-3/2} \tag{21}$$

The rate of dissipation of energy by the shock is proportional to

$$\rho_0 a^3 \left(\frac{p_1-p_0}{p_0}\right)^3 A, \tag{22}$$

the increase of entropy at the shock being proportional to the cube of the shock strength. From (13) we see that (22) also varies with s like (21); when the constants of proportionality are included, exact agreement is found.

The second point which may be considered further concerns the basic step which introduces the non-linear distortion of the wave profile, i.e. the introduction of $a+u$ for the signal speed of individual wavelets. Of course, this rests on sound physical argument, and indeed is exactly what is found ***mathematically*** in the well-known theory of simple waves in one-dimensional unsteady flow. Nevertheless, further mathematical justification is desirable. To provide this the simplified form of the results for large s ((13), (15), (17) etc.) will be established by an alternative method, which proceeds directly from the non-linear equations of motion. In fact this method gives the simplest derivation of these results.

Since the entropy jump at the shock is of third order in the strength, entropy changes may be ignored in the flow behind the shock. (This is borne out by the above considerations of energy balance.) The equations for one-dimensional flow in a tube of cross-sectional area $A(s)$ may therefore be taken as

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial s}+\rho\left(\frac{\partial u}{\partial s}+\frac{A'(s)}{A(s)}u\right) &= 0, \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial s} &= -\frac{1}{\rho}\frac{\partial p}{\partial s}, \\ p \propto \rho^{\gamma}. & \end{aligned}\right\} \tag{23}$$

Introducing the sound speed a, equations for u and a may be written

$$\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial s}+\frac{\gamma-1}{2}a\left(\frac{\partial u}{\partial s}+\frac{A'}{A}u\right) = 0, \tag{24}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial s}+\frac{2}{\gamma-1}a\frac{\partial a}{\partial s} = 0. \tag{25}$$

With the N-wave in mind, let us now consider the possibility of the following solution in series:

$$u = v_1(s)\left(t-T_0-\frac{s}{a_0}\right)+v_2(s)\left(t-T_0-\frac{s}{a_0}\right)^2+\dots, \tag{26}$$

$$a = a_0+b_1(s)\left(t-T_0-\frac{s}{a_0}\right)+b_2(s)\left(t-T_0-\frac{s}{a_0}\right)^2+\dots. \tag{27}$$

(The constant T_0 is introduced in the measurement of t, in order to agree with the previous results.) Substituting these expressions in (24) and (25) and equating coefficients of $(t-T_0-s/a_0)$, it is easily found that

$$b_1(s) = \frac{\gamma-1}{2}v_1(s) \tag{28}$$

and

$$\frac{dv_1}{ds} = \frac{\gamma+1}{2}\frac{v_1^2}{a_0^2}-\frac{A'}{2A}v_1. \tag{29}$$

Equation (29) may be written

$$\frac{d}{ds}\left(\frac{1}{v_1\sqrt{A}}\right)+\frac{\gamma+1}{2a_0}\frac{1}{\sqrt{A}} = 0;$$

hence,

$$\frac{2}{\gamma-1}b_1 = v_1 = -\frac{2a_0}{\gamma+1}\frac{1}{\sqrt{A}\int_0^s ds/\sqrt{A}}. \tag{30}$$

If only the first order terms are retained in (26) and (27), we have exactly the solution found earlier; for example,

$$\frac{p-p_0}{p_0} = \frac{2\gamma}{\gamma-1}\frac{a-a_0}{a_0} = -\frac{2\gamma}{\gamma+1}a_0\frac{t-T_0-s/a_0}{\sqrt{A}\int_0^s ds/\sqrt{A}}, \tag{31}$$

in agreement with (16) and (17). Moreover, if the head shock is specified by

$$t-\frac{s}{a_0}-T_0 = -l(s), \tag{32}$$

so that its velocity is

$$U = \left(\frac{1}{a_0} - \frac{dl}{ds}\right)^{-1} \doteqdot a_0 - a_0 \frac{dl}{ds},$$

then the shock condition (10) shows (using (31)) that

$$\frac{dl}{ds} = \frac{1}{2} \frac{l}{\sqrt{A}} \left\{ \int_0^s \frac{ds}{\sqrt{A}} \right\}^{-1}.$$

The solution of this equation is

$$l(s) \propto \left\{ \int_0^s \frac{ds}{\sqrt{A}} \right\}^{1/2}$$

in agreement with (15). The shock strength is obtained by substituting (32) in (31).

In this derivation, questions of convergence have been ignored, but it is expected that the ratio of successive coefficients in (26) and (27) decrease essentially like $1/s$ and that the series are at least asymptotically valid when $(t - T_0 - s/a_0)$ is small compared with s/a_0; $a_0(t - T_0 - s/a_0)/s$ is less than l/s, and in all the problems considered l/s tends to zero as $s \to \infty$.

The above arguments verify the results obtained for large s quite generally. Other checks on the theory will be made in specific problems by comparing the predictions of the initial shock strengths with those obtained by other methods.

3. Spherical waves

Before considering the unsymmetrical problems which are the main subject of this paper, it seems worthwhile, in order to illustrate the general theory given in § 2, to include the simple example of spherical waves.

First we note that each elementary ray tube is a cone with vertex at the centre of symmetry; therefore, if s is measured from the centre, we may choose

$$A(s) = s^2. \tag{33}$$

Secondly, we must consider the linear theory and verify that the results quoted in § 2 as being typical of all problems, are found in this case. In the theory of sound, the flow quantities may be deduced from a velocity potential ϕ which satisfies the wave equation; for spherically symmetric waves the appropriate solution is

$$\phi = \frac{f(t - s/a_0)}{s}.$$

Now, $u = \partial\phi/\partial s$, $p - p_0 = -\rho_0 \partial\phi/\partial t$; hence,

$$\frac{p - p_0}{p_0} = -\frac{\gamma}{a_0^2} \frac{f'(t - s/a_0)}{s},$$

$$u = -\frac{1}{a_0} \frac{f'(t - s/a_0)}{s} - \frac{f(t - s/a_0)}{s^2}.$$

In general the wavefront will be given by $t-(s-s_0)/a_0 = 0$, where $s = s_0$ is its position at $t = 0$; it is convenient, therefore, to introduce $\tau = t-(s-s_0)/a_0$. Then, if $F(\tau)$ is defined by

$$f(\tau) = -\frac{a_0^2}{\gamma}\int_0^\tau F(\tau')\,d\tau',$$

the expressions for $(p-p_0)/p_0$ and u become

$$\frac{p-p_0}{p_0} = \frac{F(\tau)}{s}, \tag{34}$$

$$\frac{u}{a_0} = \frac{1}{\gamma}\frac{F(\tau)}{s} + \frac{a_0}{\gamma s^2}\int_0^\tau F(\tau')\,d\tau'. \tag{35}$$

When $a_0\tau/s$ is small, the second term in (35) may be neglected in comparison with the first, and we see that the results quoted in (7) and (8) are borne out by this example.

The form of the additional term in (35) is of interest in view of earlier remarks. For, in any physically realistic case, p returns to p_0 and u returns to zero after the wave has passed. But we see that when F returns to zero after a positive phase, u still differs from zero by the (smaller) second term in (35); in order to reduce *both* p to p_0 and u to zero there must be a further negative phase so that $\int_0^\tau F(\tau')\,d\tau' \to 0$. This is an alternative argument for the existence of a negative phase to the wave. However, it is closely connected with the mass flow argument. For, to produce the large outward mass flux (proportional to $\sqrt{A} = s$) in the positive part of the wave, there must be a continual net forward transfer of mass across the wavelet $\tau = T_0$ which separates the two phases. This is represented in (34) and (35) by the fact that on $\tau = T_0$, where $p = p_0$, u is zero only to the first approximation, and the second term in (35) gives the required flux.

In any specific problem, $F(\tau)$ is determined by the boundary or initial conditions. These usually take the form of a prescribed value of u on some surface $s = R(t)$ (for example, a particle path may be prescribed), and $F(\tau)$ is obtained from (35) by solving the first order linear equation for $\int_0^\tau F(\tau')\,d\tau'$. It may be noted that in most cases the surface $R(t)$ will not be in the region where the second term of (35) can be neglected and therefore the theory of geometrical acoustics cannot be used throughout; in fact, if $R(t)$ is small the second term is dominant.

In the improved theory, there is little to add to the results in § 2; one point, however, requires care. Let us assume that the disturbance is generated at the surface of the sphere $s = R(t)$. If the initial radius $R(0)$ is not zero and $R(t)$ is approximately equal to $R(0)$, equations like (5) and (11) require only slight modification to take account of the fact that the waves start at $s = R(0)$ rather than at $s = 0$. For example, (5) becomes

$$t = \frac{s-R(0)}{a_0} - kF(\tau)\int_{R(0)}^{s}\frac{ds}{\sqrt{A}} + \tau. \tag{36}$$

But, if $R(0) = 0$, more care is required since with $A = s^2$, $\int ds/\sqrt{A}$ is not convergent at $s = 0$. However, the lower limit is chosen so that τ agrees approximately with its linear value for the initial propagation of the wave form; hence, we may, with greater accuracy, replace $R(0)$ in (36) by $R(\tau)$, where $R(\tau)$ is the value of R when the wavelet labelled by τ is generated at the sphere. Then, using $A(s) = s^2$, τ is determined from

$$t = \frac{s-R(\tau)}{a_0} - kF(\tau)\log\frac{s}{R(\tau)} + \tau. \tag{37}$$

Similarly, the relation for $T(s)$, the value of τ at the shock, becomes

$$\log\frac{s}{R(T)} = \frac{2}{k}\frac{\int_0^T F(T')\,dT'}{F^2(T)}. \tag{38}$$

To provide a check on the theory, we may consider the shock produced by a sphere expanding at a uniform rate, since this special case has been solved using a different method by Lighthill (1948). If the radius of the sphere at time t is $R(t) = ma_0t$, the boundary condition is

$$u = ma_0 \quad \text{on} \quad R = ma_0t.$$

From (35), this determines $F(\tau)$ for small values of m to be

$$F(\tau) = 2\gamma m^3 a_0 \tau.$$

Thus, since $F(\tau)$ is continuous at $\tau = 0$, there is no pressure jump according to linear theory. But, in the improved theory a shock is predicted, as required, although its strength is extremely small. Equation (38) gives

$$\log\frac{s}{ma_0T} = \frac{1}{k2\gamma m^3 a_0} = \frac{1}{(\gamma+1)m^3}.$$

Therefore, at the shock,

$$\frac{p_1-p_0}{p_0} = \frac{F(T)}{s} = 2\gamma m^2 e^{-(\gamma+1)^{-1}m^{-3}},$$

and the shock velocity is given by

$$\frac{U}{a_0} - 1 = \frac{\gamma+1}{2}m^2 e^{-(\gamma+1)^{-1}m^{-3}}.$$

In these expressions, the factors multiplying the exponential are suspect, since the error terms in the exponential may well be more important. The most that can be said with certainty is that

$$\log\left(\frac{U}{a_0}-1\right) \sim \frac{1}{(\gamma+1)m^3}.$$

This is the result obtained by Lighthill.

4. Unsymmetrical explosions

In this section the theory is applied in detail to the explosion model described in § 1. We consider a high pressure region V of arbitrary shape in which the gas is at rest at a pressure $p_0 + P$, and which is surrounded by an infinite expanse of gas at rest with pressure p_0; at time $t = 0$, the high pressure region is released. Since only weak shocks are considered in this theory, it is assumed that P/p_0 is small. Furthermore it will be assumed that the region V is convex; the difficulties which arise when this is not the case will be noted later.

The rays in this case are normals to the surface of V, and the expression for $A(s)$ is easily found in terms of the curvature of the surface at the foot of the ray. If R_1 and R_2 denote the principal radii of curvature at any point P of the surface, the radii of curvature of the wavefront at a distance s out along the ray from P are R_1+s and R_2+s. Therefore, by considering the area of a small curvilinear rectangle formed by the principal curves on the surface, it is seen that the area of any small element of the wavefront is proportional to $(R_1+s)(R_2+s)$. Thus, we may take

$$A(s) = (R_1+s)(R_2+s). \tag{39}$$

We next verify the form of the solution (7) and (8) and determine the function F from the full linear theory. In the linearized formulation of the problem the velocity potential $\phi(x,y,z,t)$ satisfies the wave equation,

$$\nabla^2\phi = \frac{1}{a_0^2}\frac{\partial^2\phi}{\partial t^2},$$

and the initial conditions

$$\phi = 0, \qquad \frac{\partial\phi}{\partial t} = \begin{cases} -\dfrac{P}{\rho_0} \text{ in } V, \\ 0 \text{ outside } V, \end{cases}$$

where ρ_0 is the density of the gas. (The first condition arises since the velocity $v = \nabla\phi$ is zero, and the second from the result that in the theory of sound the pressure excess $p-p_0$ is given by $-\rho_0\partial\phi/\partial t$.) This is a special case of a classical problem solved by Poisson and the derivation of the solution is given, for example, in Lamb (1932, Art. 287). The solution for general initial values of ϕ and $\partial\phi/\partial t$ is

$$\phi(x,y,z,t) = tM_{a_0t}\left[\frac{\partial\phi}{\partial t}\right] + \frac{\partial}{\partial t}\left\{tM_{a_0t}[\phi]\right\},$$

where $M_{a_0t}[\phi]$ and $M_{a_0t}[\partial\phi/\partial t]$ represent the mean values of the initial values of ϕ and $\partial\phi/\partial t$ taken over the surface of a sphere with centre at (x,y,z) and radius a_0t. Thus, in the present case,

$$\phi = -\frac{P}{4\pi\rho_0 a^2}\frac{g(x,y,z,t)}{t}, \tag{40}$$

where $g(x,y,z,t)$ is the area of the part of the surface of the sphere which lies inside V.

We now consider the variation of ϕ with t at a point a distance s along the ray from the point P on the surface of V. In particular, we consider the approximate form of the solution near the head of the wave and for large values of s. Clearly g remains zero until the radius a_0t of the sphere reaches s, corresponding to the arrival of the wavefront at $t = s/a_0$; we set $\tau = t-s/a_0$.

For small τ, the sphere will intersect V in a small curve surrounding P. If we choose coordinates (ξ, η, ζ) with origin at P, ζ measured along the ray at P and ξ and η in the principal directions at P, the equation of the surface of V in the neighbourhood of P is approximately

$$\zeta = -\frac{\xi^2}{2R_1} - \frac{\eta^2}{2R_2}. \tag{41}$$

The equation of the sphere is

$$(\zeta - s)^2 + \xi^2 + \eta^2 = (s + a_0\tau)^2, \tag{42}$$

and, assuming that $a_0\tau/s$ is small, its intersection with (41) satisfies

$$\frac{\xi^2}{2a_0\tau}\left(\frac{1}{R_1} + \frac{1}{s}\right) + \frac{\eta^2}{2a_0\tau}\left(\frac{1}{R_2} + \frac{1}{s}\right) = 1. \tag{43}$$

To the same approximation, the area of the surface of the sphere bounded by this curve is equal to its projection on the plane $\zeta = 0$, i.e. it is the area of the ellipse (43). Therefore,

$$g(x, y, z, t) = 2\pi a_0\tau\left\{\frac{R_1R_2s^2}{(R_1+s)(R_2+s)}\right\}^{1/2}. \tag{44}$$

Then, setting $t \doteqdot s/a_0$ in the denominator in (40), we have

$$\phi = -\frac{P}{2\rho_0}\left\{\frac{R_1R_2}{(R_1+s)(R_2+s)}\right\}^{1/2}\tau, \quad \text{for small } \tau. \tag{45}$$

It may be noted that $p - p_0 = -\rho_0\partial\phi/\partial t$ jumps discontinuously from 0 to $\frac{1}{2}P$ at $s = 0$, and this agrees with the well-known one-dimensional result.

The solution is also required when τ is not small but s is large. For sufficiently large s, it is clear that the part of the sphere intercepted by V may be approximated by a plane perpendicular to the ray at P and at a distance $a_0\tau$ along the ray inside V. The area of this plane inside V is independent of s. Therefore, again taking $t \doteqdot s/a_0$ in (40), we have

$$\phi = -\frac{P}{4\pi\rho_0a_0}\frac{f(\tau)}{s}, \tag{46}$$

where $f(\tau)$ is the cross-sectional area of V at a distance $a_0\tau$ along the inside normal from the point P, the section being taken perpendicular to the normal. Only the dependence of f on τ is shown, but it must be remembered that it also varies from ray to ray. To the same order of approximation, we may write

$$\phi = -\frac{P}{4\pi\rho_0a_0}\frac{f(\tau)}{\sqrt{[(R_1+s)(R_2+s)]}}, \tag{47}$$

which brings the result into the same form as (45). In fact (45) may be included in (47), provided that $f(\tau) \sim 2\pi a_0\tau\sqrt{(R_1R_2)}$ for small τ. But this is so; the intersection of $\zeta = -a_0\tau$ with (41) is an ellipse with semiaxes $\sqrt{(2a_0\tau R_1)}$, $\sqrt{(2a_0\tau R_2)}$, and the result follows. Hence, (47) applies both for small τ and large s, and the appropriate requirement is that $a_0\tau/s$ is small.

As noted in (38), $A(s) \propto (R_1+s)(R_2+s)$; thus, (47) is of the form predicted by geometrical acoustics. We have

$$\frac{p-p_0}{p_0} = -\frac{\rho_0}{p_0}\frac{\partial\phi}{\partial t} = \frac{F(\tau)}{\sqrt{A}}, \tag{48}$$

where

$$F(\tau) = \frac{P}{\rho_0}\frac{f'(\tau)}{4\pi a_0}, \tag{49}$$

in accordance with (7). Moreover, the particle velocity, $\partial\phi/\partial s$, along the ray agrees with (8). Thus, all the results quoted in the general theory of §§ 1 & 2 are illustrated in this example. The non-linear theory derived in § 2 applies directly to this problem, with the functions $A(s)$ and $F(\tau)$ for each ray given by (38) and (49), respectively. In the theory, $\int_0^s A^{-1/2}\,ds$ is required, and we may note that it is given by

$$\int_0^s \frac{ds}{\sqrt{A}} = 2\log\left\{\frac{\sqrt{(R_1+s)}+\sqrt{(R_2+s)}}{\sqrt{R_1}+\sqrt{R_2}}\right\}. \tag{50}$$

The shock strength is given as a function of s by (48) with $\tau = T(s)$ as determined by (11).

Since $F(\tau)$ tends to the finite value $\frac{1}{2}P/p_0$ as $\tau \to 0$ the initial strength of the shock is $\frac{1}{2}P/p_0$ as in the corresponding problem of one-dimensional flow; the compression wave increasing p from p_0 to $p_0+\frac{1}{2}P$ travels out into the undisturbed medium and an expansion wave reducing the pressure from p_0+P to $p_0+\frac{1}{2}P$ travels into the high pressure region. Then, for large values of s, the simple law (13) applies, and A is proportional to s^2 so that the variation of shock strength with s is just as for a spherical wave. But, the constant of proportionality in (13) varies with the direction of the ray. It is seen from (49) that the zero T_0 of $F(\tau)$ corresponds to the maximum cross-sectional area of V perpendicular to the ray considered. The function $f(\tau)$ increases from zero to a maximum at $\tau = T_0$ and then decreases to zero; hence $F(\tau)$ takes both positive and negative values and again

$$\int_0^\infty F(\tau)\,d\tau = 0.$$

From (49),

$$\int_0^{T_0} F(\tau)\,d\tau = \frac{P}{4\pi a_0 p_0} f(T_0);$$

hence, (13) gives

$$\frac{p_1-p_0}{p_0} \sim \left\{\frac{\gamma}{\gamma+1}\frac{1}{\pi}\frac{P}{p_0}f(T_0)\right\}^{1/2} s^{-1}\left\{\log\frac{4s}{(\sqrt{R_1}+\sqrt{R_2})^2}\right\}^{-1/2}, \tag{51}$$

for large s. The shock strength at a fixed distance is proportional to the square root of the excess pressure and to the square root of the maximum cross-sectional area of V for that direction. Thus the shock is strongest in the directions for which the projected area of V is greatest. Even for strong explosions, we may expect the predictions of the theory to be qualitatively correct, and the directional variation of shock strength to be correlated with the projected areas of V.

It is perhaps worth noting that for the special case in which V is a sphere of radius R_0, (47) is the exact solution for all s and τ. In this case

$$f(\tau) = \pi(2R_0 - a_0\tau)a_0\tau, \qquad 0 < a_0\tau < 2R_0,$$

so that

$$F(\tau) = \frac{P}{2p_0}(R_0 - a_0\tau), \qquad 0 < a_0\tau < 2R_0.$$

Thus, the disturbance is an N-wave from its inception, not only at distances sufficiently large for (16) to apply.

If the boundary of V includes a region which is concave outwards, the rays will be distorted as shown in figure 2; but, after this region has been left behind, the rays will diverge, and ultimately the shock will follow the usual $s^{-1}(\log s)^{-1/2}$ law of decay. Equation (51) will probably still give a reasonably accurate form for the constant of proportionality.

It should be noted that even if the boundary is not concave but is plane over some small region distortion of the rays is still required to obtain an accurate result. For the ray tube area $A(s)$ is constant on the rays from a plane region, and the propagation along these rays is initially similar to the case of plane waves. Hence the shock will not decay with s as rapidly as on the neighbouring diverging rays. Now this is the essential point: the dependence on s follows a different law so that if unchecked the relative difference in shock strength becomes large as $s \to \infty$. But, starting first at the edge of the plane region, the effect of the large difference in shock strength will be to curve the rays outwards, and, eventually, they all diverge giving decay of the spherical type. It is interesting to observe that this view is confirmed by the expressions found for ϕ: (45) applies for small τ and gives the plane wave formula as R_1 and R_2 tend to infinity, but (46) still holds for large s and gives the spherical form for ϕ.

When concave regions of the boundary V are admitted, the function $f(\tau)$ may have more than one maximum for some directions. If this is so, there will be compression regions in which $F'(\tau) > 0$ in addition to the two main compressions, and this leads to the formation of additional shocks. These may be determined in the same way as the multiple tail shocks in the problem of the supersonic projectile (Whitham 1952), but ultimately they run into one of the two main shocks.

5. Supersonic bangs

An important application of the general theory is to the determination of the shocks produced by a body in non-uniform supersonic flight. This application has been developed in detail by P. Sambasiva Rao (1956 a & b), and the main results are quoted here for completeness.

Near the body, just as in the case of uniform supersonic motion, the wavefront forms a cone about the direction of flight with semi-angle equal to the Mach angle $\sin^{-1} 1/M$, where M is the Mach number of the body at that point. Hence, the rays from any point of the flight path initially make an angle $\cos^{-1} 1/M$ with the direction of motion. But if the medium is assumed to be uniform, the rays must remain straight. Hence, we have the typical pattern shown in figure 4 for an accelerating source. If the velocity were constant, the area of each ray tube would be increasing proportional to s, due to the cylindrical spreading of the wavefront away from

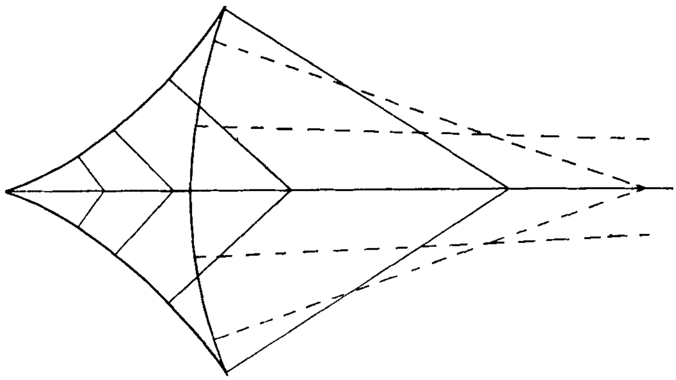

Figure 4.

the axis. But if the body accelerates, $\cos^{-1} 1/M$ increases and the rays in any meridian plane converge; conversely, if the body decelerates, the rays diverge. It is easily shown that the modification due to this extra convergence or divergence is included by taking

$$A(s) = s\left(1 - \frac{s}{\lambda}\right), \tag{52}$$

where

$$\lambda = \frac{a_0(M^2-1)}{M^{-1}dM/dt}, \tag{53}$$

M and dM/dt referring to the Mach number of the body when it is at the foot of the ray concerned. For acceleration, $\lambda > 0$, and we see that A vanishes when $s = \lambda$; at this point the wavefront is cusped. In figure 4, the rays shown as full lines 'carry' the forward part of the wavefront and $s < \lambda$ on each of these rays; the rays shown as broken lines 'carry' the rear part of the wavefront and on each of these s has exceeded λ. For a curved flight path Rao (1956) shows that $A(s)$ is still given by (52) but λ is modified. The denominator of λ is the component along the ray of the acceleration of the body; for a curved path this includes a term due to the transverse acceleration of the body.

If the acceleration of the body is relatively small (the change of velocity in flying a body length is small, say) we might expect the initial wave profile to be the same as in the case of uniform motion; Rao's detailed investigations confirm this. Therefore, the F-function is the same as in Whitham (1952). Introducing a constant of proportionality in order that F conforms to the definition (7), it is found that

$$F(\tau) = \frac{1}{\gamma}\left\{\frac{M^5}{2(M^2-1)}\right\}^{1/2} \frac{1}{2\pi} \int_0^{U\tau} \frac{S''(\xi)\, d\xi}{\sqrt{(U\tau-\xi)}}, \tag{54}$$

where $S(\xi)$ is the cross-sectional area of the body at a distance ξ from the nose, and as usual, τ measures the time after the wavefront passed the point.

With $A(s)$ given by (52) and $F(\tau)$ by (54), the general results of § 2 may be used. For $s \ll \lambda$, $A(s) \sim s$ and the shock spreads away from the axis like a cylindrical shock. If the velocity of the body is uniform, λ is infinite and this behaviour extends right to infinity with the ultimate decay in shock strength like $s^{-3/4}$. But if λ is finite, we see that for sufficiently large s, $A \propto s^2$ so that the decay is eventually like a spherical shock with strength proportional to $s^{-1}(\log s)^{-1/2}$. It should be remembered, however, that when $\lambda > 0$, a cusp will have intervened before this region is reached, and in the neighbourhood of the cusp the simple theory of geometrical acoustics breaks down. Hence, we are assuming that beyond the cusp the theory may again be applied, at least so far as the main features are concerned. The question of what happens near a cusp and what effects it has on the results is discussed in more detail in Rao (1956 a & b).

Here, only the highlights of the theory of supersonic bangs have been mentioned in order to show the generality of the theory developed in § 2; for the details and practical predictions, reference should be made to Rao's papers.

6. Steady supersonic flow: an example in cone theory

The theory developed in the first part of the paper can be interpreted and applied in certain problems of steady supersonic flow past unsymmetrical bodies. If we introduce cylindrical polar co-ordinates (r, θ, x) and let U be the velocity of the undisturbed stream, then the steady flow problem is analogous to an unsteady flow in the (r, θ) plane with x/U playing the role of time. Thus, for a pointed body, the analogy in unsteady flow would be to the disturbance produced by a solid cylinder which starts with radius zero and expands with arbitrary shape. If the body is swept behind the Mach cone through the nose, the corresponding rate of expansion of the cylinder is subsonic. In that case, the rays would be straight lines through the origin $r = 0$, and the linear theory would predict amplitudes proportional to $r^{1/2}$. Hence, for the steady problem, the analogue to treating the propagation in each ray tube separately is to consider the flow in each meridian plane $\theta =$ constant separately. Near the Mach cone, which is the analogue of the wavefront, the amplitude of the disturbance will be proportional to $r^{-1/2}$. Thus any dependence on θ must be introduced by the profile function F.

If we consider a body which is everywhere slender (i.e. its slope in the stream direction is always small) and whose cross-section has no regions of abnormally large curvature, the linear theory developed by Ward (1949) shows that, although the flow near the body varies with θ, the F-function giving the flow near the Mach cone is independent of θ. Hence, the shock is the same as for a body of revolution with the same distribution of cross-sectional area. In order to obtain an example involving an unsymmetrical shock, we must relax these conditions on the body shape.

We consider the flow past a flat plate delta wing at incidence, the edges of the wing being swept back well behind the Mach cone. This is a problem in which there is no fundamental length and hence the velocity components must be equal to the main stream velocity U multiplied by functions of r/x and θ; the velocity potential has an additional factor x since it has an extra dimension of length. This is a so-called cone field problem, and the linearized solution is known. Furthermore, Lighthill (1949) has shown how the shock strengths may also be found in such cone field problems. Here, the results are derived by the general theory which does not rely on the special properties of cone fields; the agreement with Lighthill's result gives independent confirmation of the assumptions of the theory.

A full account of the linear theory is given by Goldstein and Ward (1950). The velocity potential is taken as $U(x+\phi)$ where $\phi = xf(Br/x, \theta)$, $B = \sqrt{(M^2-1)}$, and it may be shown for the present problem that, near the Mach cone $x = Br$,

$$f \sim -\tfrac{2}{3}g(\theta)\left(1-\frac{Br}{x}\right)^{3/2}. \tag{55}$$

The function $g(\theta)$ introduced here is identical with the $A(\theta)$ used by Lighthill, and is given by

$$g(\theta) = -\frac{L\alpha\sqrt{2}\sin\theta}{B(1-t_0\cos\theta)^{3/2}(1-t_1\cos\theta)^{3/2}}, \tag{56}$$

where θ is measured from the plane of the wing, α is the angle of incidence of the wing, $\cot^{-1} t_0/B$ and $\cot^{-1} t_1/B$ are the angles made with the x axis by the leading edges of the wing, and

$$L = \frac{(t_0-t_1)^2}{2(1+t_0)^{1/2}(1-t_1)^{1/2}\{2E(k)-(1-k^2)K(k)\}},$$

$$k^2 = \frac{(1+t_1)(1-t_0)}{(1-t_1)(1+t_0)},$$

where $E(k)$ and $K(k)$ are the complete elliptic integrals.

From (55), we see that

$$\phi = -\frac{2}{3}\frac{g(\theta)}{(Br)^{1/2}}(x-Br)^{3/2} \tag{57}$$

for the flow near the Mach cone, and it may be noted that, as in previous examples, the crucial condition is that τ/r should be small, where the characteristic variable τ is now $x-Br$. Moreover, we see that the azimuthal component of the perturbation velocity, $U\phi_\theta/r$, is of order τ/r times the other components $U\phi_r U\phi_x$. Thus, to a first approximation, the flow is in meridian planes and θ plays the role of a parameter. This is in accordance with the general theory. From (57), we have

$$\phi_x = -\frac{g(\theta)(x-Br)^{1/2}}{(Br)^{1/2}},$$

$$\phi_r = \frac{B^{1/2}g(\theta)(x-Br)^{1/2}}{r^{1/2}},$$

or, introducing a standard notation,

$$\left.\begin{aligned}\phi_x &= -\frac{F(\tau)}{(2Br)^{1/2}},\\ \phi_r &= -\frac{BF(\tau)}{(2Br)^{1/2}},\end{aligned}\right\} \tag{58}$$

where

$$F(\tau) = g(\theta)2^{-1/2}\tau^{1/2}. \tag{59}$$

The improved non-linear theory is obtained by finding a more accurate relation for the characteristic variable $\tau(x, r)$, in place of $\tau = x - Br$. This procedure is quite straightforward, and is the same as the corresponding step in the axisymmetrical problem (Whitham 1952) since θ appears only as a parameter. Briefly, it is as follows. The characteristic direction at any point makes the local Mach angle μ with the stream direction χ. Hence, τ is constant on each curve satisfying

$$\frac{dx}{dr} = \cot(\mu + \chi).$$

The sound speed is given in terms of ϕ by Bernoulli's equation; hence $\cot(\mu + \chi)$ can be expressed in terms of ϕ. Using $\phi_r = -B\phi_x$, it is found that τ is to be determined by

$$\left(\frac{dx}{dr}\right)_{\tau = \text{constant}} = B + \frac{\gamma+1}{2B}M^4\phi_x,$$

to the first order in ϕ. Therefore, from (58), the relation for τ is

$$x = Br - k_1 F(\tau) r^{1/2} + \tau, \tag{60}$$

where

$$k_1 = \frac{(\gamma+1)M^4}{2^{1/2}B^{3/2}}. \tag{61}$$

When $F(\tau) < 0$, the characteristics are diverging in an expansion wave; but when $F(\tau) > 0$, they converge and form a shock. In the latter case the shock can be determined by giving the value $T(r)$ of τ at points just behind the shock, and if the flow ahead of the shock is undisturbed, the condition is

$$r^{1/2} = \frac{2\int_0^T F(T')dT'}{k_1 F^2(T)}. \tag{62}$$

The derivation of this result from (60) is identical with that given in Whitham (1952); it also follows closely the derivation of (11) from (5). The equation of the shock is then given by substituting $\tau = T(r)$ in (6).

From Bernoulli's equation, $p - p_0 \doteqdot -\rho_0 U^2\phi_x$ where p_0, ρ_0 are the pressure and density in the main stream; hence,

$$\frac{p - p_0}{p_0} = -\gamma M^2 \phi_x = \frac{\gamma M^2 F(\tau)}{(2Br)^{1/2}}. \tag{63}$$

The shock strength is found by substituting $\tau = T(r)$ in this expression.

For the problem of the delta wing, $F(\tau) < 0$ above the wing where $0 < \theta < \pi$, and there is an expansion wave as expected. But below the wing $F(\tau) > 0$ and there is a shock. In this case, (62) gives

$$T = \frac{9k_1^2}{32} g^2(\theta) r,$$

so that

$$F(T) = \tfrac{3}{8} k_1 g^2(\theta) r^{1/2}.$$

Hence, the equation of the shock is

$$x = Br - \frac{3}{16} \frac{(\gamma+1)^2 M^8}{B^3} g^2(\theta) r, \tag{64}$$

and its strength is

$$\frac{p_1 - p_0}{p_0} = \tfrac{3}{4}\gamma(\gamma+1)\frac{M^6}{B} g^2(\theta). \tag{65}$$

These expressions agree exactly with Lighthill's results.

7. Thin wings of finite span

In the previous example only the profile function F depended on the orientation of the plane in which the flow was considered. We now turn to a problem in which the amplitude function (which was merely $r^{-1/2}$ in the last example) also varies with orientation. This is the case in supersonic flow past a wing with planform of the general shape shown in figure 5.

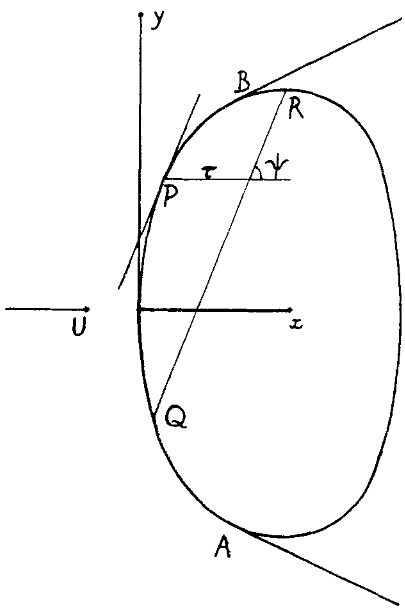

Figure 5.

The linearized theory has been developed in detail (see, for example, the account in Ward (1955)) so that all the information required for applying the theory of this paper is obtainable.

First we consider the geometry of the wavefront in order to deduce the set of planes in which the flow should be considered and to obtain the amplitude of the disturbances near the wavefront. We then turn to the linearized solution for the profile function F and, incidentally, for verification of the geometrical prediction of the amplitude.

For simplicity, we consider only the problem in which the flow is symmetric relative to the mean plane of the wing; this is sometimes called the 'thickness problem'. It is assumed that the boundary of the wing lies in the (x, y) plane with the x axis in the direction of the main stream, that the upper and lower surfaces of the wing are given by

$$z = \pm Z(x, y), \tag{66}$$

and that the boundary of the wing planform is given by

$$x = l(y). \tag{67}$$

The wavefront is the envelope of the Mach cones with vertices on the supersonic leading edge AB of the wing, A and B being the points where the boundary makes the Mach angle μ with the main stream. If $(l(\eta), \eta)$ is a point on AB, the Mach cone is

$$(x - l(\eta))^2 = B^2(y-\eta)^2 + B^2z^2. \tag{68}$$

At points on the envelope the derivative of (68) with respect to η is also satisfied, so that

$$(x - l(\eta))l'(\eta) = B^2(y-\eta). \tag{69}$$

The latter is the equation of a plane through $(l(\eta), \eta)$ parallel to the z axis and making an angle $\tan^{-1} l'(\eta)/B^2$ with the x axis. This plane is shown as PNM in figure 6, where P is $(l(\eta), \eta)$ and PM is the intersection of the

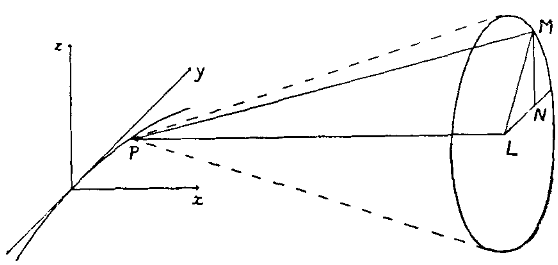

Figure 6.

plane with the cone (68); PL is the stream direction and PN is the intersection of the plane (69) with $z = 0$. The wavefront is the surface generated by PM as P moves along AB. Now in applying the theory, the flow is considered separately in each normal plane PLM. The coordinates corresponding to the ray coordinates of the unsteady flow problems are

x, η and r, where η specifies the plane PLM and r measures distance from PL in this plane. Since $L\hat{P}N = \tan^{-1}l'(\eta)/B$ and $M\hat{P}L = \mu = \cot^{-1}B$, it follows that $\theta = M\hat{L}N$ is given by

$$\cos\theta = \frac{l'(\eta)}{B}. \tag{70}$$

Hence, the Cartesian coordinates (x, y, z) are related to (x, η, r) by

$$\left.\begin{aligned} x &= x, \\ y &= \eta + r\cos\theta, \\ z &= r\sin\theta. \end{aligned}\right\} \tag{71}$$

The function corresponding to the area $A(s)$ of a ray tube, is now the distance between two neighbouring normal planes PLM. If we consider the two planes corresponding to η and $\eta + \delta\eta$ and denote this distance by $D(r)\delta\eta$, the initial value of D is $\sin\theta$; therefore, since the planes diverge with angle $\delta\theta$, $D(r)$ is given by

$$\begin{aligned} D(r) &= \sin\theta - r\frac{d\theta}{d\eta}, \\ &= \frac{\sqrt{(B^2 - l'^2)}}{B} + \frac{l''}{\sqrt{(B^2 - l'^2)}}r. \end{aligned} \tag{72}$$

The amplitudes of the disturbances vary like $D^{-1/2}$.

In order to find the solution near the wavefront in detail, we must turn to the full linear theory. But, we can already predict that the velocity potential ϕ will take the form

$$\phi = \frac{-f(\tau)}{\sqrt{[2B(r + (B^2 - l'^2)/Bl'')]}}, \tag{73}$$

where τ, which measures distance behind the wavefront in a streamwise direction, is given by

$$\tau = x - l(\eta) - Br. \tag{74}$$

(Although only the dependence of f on τ is shown explicitly it should be remembered that f is also a function of η.) It is convenient to take the amplitude as given in (73) rather than as $D^{-1/2}(r)$ itself, but it should be noted that this assumes $l''(\eta) \neq 0$ at any point of the leading edge of the wing. The situation when $l''(\eta) = 0$ is very similar to the case of zero curvature of the surface in the problem of § 4. In this case, the normal planes are parallel and the shock initially behaves as in the essentially two-dimensional problem of a swept back wing of constant section. However, as r increases, the normal planes will in reality curve away from each other to give the eventual decay typical of a finite body.

For the symmetrical problem, the complete linearized solution is

$$\phi(x, y, z) = -\frac{1}{\pi}\iint \frac{Z_1(x', y')\, dx'dy'}{\sqrt{[(x - x')^2 - B^2(y - y')^2 - B^2z^2]}}, \tag{75}$$

where $Z_1(x,y) = \partial Z(x,y)/\partial x$, and the integration is over the part of the wing in the region

$$x - x' \geqslant B\sqrt{[(y-y')^2 + z^2]}; \tag{76}$$

this region of integration is the intersection with the plane $z = 0$ of the *upstream* Mach cone from (x, y, z).

The approximate form of (75) is required both for small τ and large r. These results turn out to be closely similar to the corresponding ones in § 4, and the details of the check on (73) for small τ is omitted. It is found that

$$f(\tau) \sim \frac{\sqrt{2\epsilon(\eta)\tau}}{\sqrt{l''(\eta)}} \quad \text{for small } \tau, \tag{77}$$

where $\epsilon(\eta) = Z_1(l(\eta), \eta)$ is the slope of surface in the x-direction at the leading edge. The approximation of (75) for large r is now obtained, and it gives a function $f(\tau)$ which confirms (77). We set $x' = l(\eta) + \alpha$, and $y' = \eta + \beta$, in the double integral (75); then, since the variations of α and β are small compared to r, we have approximately

$$\begin{aligned}(x-x')^2 - B^2(y-y')^2 - B^2z^2 \\ &= (Br+\tau+\alpha)^2 - B^2\left(\frac{l'r}{B} - \beta\right)^2 - B^2r^2\left(1 - \frac{l'^2}{B^2}\right) \\ &\doteqdot 2Br(\tau - \alpha + l'(\eta)\beta).\end{aligned}$$

Thus,

$$\phi = -\frac{1}{\sqrt{2Br}}\frac{1}{\pi}\iint\frac{Z_1\,d\alpha d\beta}{\sqrt{[\tau-(\alpha-l'\beta)]}}, \tag{78}$$

and the region of integration is bounded by the straight line $\alpha - l'\beta = \tau$ which is parallel to the tangent to the leading edge at P and is at a distance τ downstream from P (see QR in figure 5). Hence, the integral in (89) is independent of r, and comparing (78) with (73) it is seen that

$$f(\tau) = \frac{1}{\pi}\iint\frac{Z_1\,d\alpha d\beta}{\sqrt{[\tau-(\alpha-l'\beta)]}}. \tag{79}$$

The expression (79) can be written in a form which agrees exactly with the corresponding result in flow past a body of revolution. The region of integration is divided into elementary sections parallel to QR, then $\alpha - l'\beta$ is constant and equal to t, say, on any section. The thickness of an elementary section is $\sin\psi\,dt$ where ψ is the angle between QR and the stream direction; hence, for this slice,

$$\iint 2Z\,d\alpha d\beta = S^*(t)\sin\psi\,dt = S(t)\,dt$$

where $S^*(t)$ is the cross-sectional area of the slice and $S(t)$ denotes the projection of the area $S^*(t)$ perpendicular to the stream. Thus, (79) becomes

$$f(\tau) = \frac{1}{2\pi}\int_0^\tau \frac{S'(t)\,dt}{\sqrt{(\tau-t)}}, \tag{80}$$

and

$$\phi = -\frac{f(\tau)}{\sqrt{2Br}}, \tag{81}$$

X 2

for large r. This result is exactly the same as the velocity potential at distance r from the axis in flow past a body of revolution whose cross-sectional area at distance t from the nose is $S(t)$. Thus, *in each plane normal to the wavefront*, the wing can be replaced by an equivalent body of revolution (as far as the flow at large distances is concerned). Of course, the equivalent body of revolution is not the same for each plane.

The determination of the shock etc. is formally the same as in § 6 with the slight modification that, in view of (73), r must be replaced by $r + r_0(\eta)$ where

$$r_0(\eta) = \frac{B^2 - l'^2}{Bl''}.$$

The velocity components and the pressure are given by

$$\phi_x = -\frac{F(\tau)}{\sqrt{[2B(r+r_0)]}}, \quad \phi_r = \frac{BF(\tau)}{\sqrt{[2B(r+r_0)]}}, \tag{82}$$

$$\frac{p - p_0}{p_0} = \frac{\gamma M^2 F(\tau)}{\sqrt{[2B(r+r_0)]}},$$

where

$$F(\tau) = f'(\tau) = \frac{1}{2\pi}\int_0^\tau \frac{S''(t)\,dt}{\sqrt{(\tau - t)}}. \tag{83}$$

The corrected expression to determine τ is

$$x = l(\eta) + Br - \frac{(\gamma+1)M^4}{2^{1/2}B^{3/2}} F(\tau)(r + r_0)^{1/2} + \tau. \tag{84}$$

At the shock, $\tau = T(r)$ where

$$(r + r_0)^{1/2} - r_0^{1/2} = \frac{(2B)^{3/2}}{(\gamma+1)M^4} \frac{\int_0^T F(\tau)\,dt}{F^2(T)}. \tag{85}$$

The initial strength is given by setting $T = 0$ in (82); hence, since $F(0) = f'(0) = \sqrt{2}\epsilon(\eta)/\sqrt{l''}$ (see (77)), it is

$$\frac{\epsilon(\eta)\gamma M^2}{\sqrt{[B^2 - l'^2(\eta)]}}. \tag{86}$$

This is the same as the linear result, of course, and agrees also with the result for flow past a wing of uniform section and initial slope ϵ, swept back through an angle $\tan^{-1} l'(\eta)$.

For large r, the law of decay is found by taking T near to the zero T_0 of $F(\tau)$, so that (85) gives the approximation

$$F(T) \sim \left\{\frac{(2B)^{3/2}}{(\gamma+1)M^4}\int_0^{T_0} F(\tau)\,d\tau\right\}^{1/2} r^{-1/4}. \tag{87}$$

Then, from (82), we obtain

$$\frac{p_1 - p_0}{p_0} \sim \frac{\gamma}{(\gamma+1)^{1/2}} \frac{(2B)^{1/4}}{r^{3/4}}\left\{\int_0^{T_0} F(\tau)\,d\tau\right\}^{1/2}. \tag{88}$$

The shock decays in each normal plane exactly as in the meridian plane for a body of revolution having the appropriate F-curve, and the directional variation of strength arises only from the dependence of F on η.

8. Note on the wave drag of a finite wing

The wave drag on a body can be calculated from the rate of dissipation of energy by the shocks. For a body of revolution the drag is then obtained in the form

$$\pi\rho_0 U^2 \int_0^\infty F^2(\tau)\,d\tau, \tag{89}$$

(see Whitham 1952), where $F(\tau)$ is the F-function for the body. The drag on the wing can be calculated in a similar way.

In the first instance, the contribution to the drag from the shock attached to the leading edge of the wing is

$$2\int_0^\infty \int_{\eta_1}^{\eta_2} \rho_0 T_0 s D(r)\,dr d\eta, \tag{90}$$

where ρ_0 and T_0 are the density and temperature of the main stream, s is the entropy jump at the shock, $D(r)$ is the function defined by (72) and $\eta_1 \leqslant \eta \leqslant \eta_2$ defines the leading edge of the wing. Now

$$T_0 s = \frac{(\gamma+1)U^2}{12\gamma^3 M^2}\left(\frac{p_1-p_0}{p_0}\right)^3,$$

and

$$D(r) = -[r+r_0(\eta)]\frac{d\theta}{d\eta}.$$

Hence substituting the value of $(p_1-p_0)/p_0$ from (82), the expression (90) becomes

$$\rho_0 U^2 \int_0^\infty \int_0^\pi \frac{(\gamma+1)M^4}{6(2B)^{3/2}} F^3(T)(r+r_0)^{-1/2}\,dr d\theta.$$

When r is eliminated by means of (85), this gives

$$\rho_0 U^2 \int_0^{T_0} \int_0^\pi \tfrac{1}{3}F^3(T)\frac{d}{dT}\left\{\frac{\int_0^T F(\tau)\,d\tau}{F^2(T)}\right\} dT d\theta$$

$$= \rho_0 U^2 \int_0^\pi \left(\int_0^{T_0} F^2(T)\,dT\right) d\theta. \tag{91}$$

Although the shocks from the trailing edge have not been treated here, it is reasonable to assume that they will contribute the terms $\int_{T_0}^\infty F^2(T)\,dT$, since this is the contribution of the tail shock in the body of revolution case. If this is assumed the drag on the wing becomes

$$\rho_0 U^2 \int_0^\pi \left(\int_0^\infty F^2(T)\,dT\right) d\theta. \tag{92}$$

On comparing this result with (89), it is seen that *the drag on the wing is the mean of the drags on the equivalent bodies of revolution introduced in Section* 7. Of course, the expression (92) for the drag must be equal to the value obtained by the more direct evaluation from the pressures on the wing surface, but for some purposes this may be a more useful form. It is certainly a much more significant form for the present theory.

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